

POST-LIE ALGEBRA STRUCTURES AND GENERALIZED DERIVATIONS OF SEMISIMPLE LIE ALGEBRAS

DIETRICH BURDE AND KAREL DEKIMPE

ABSTRACT. We study post-Lie algebra structures on pairs of Lie algebras $(\mathfrak{g}, \mathfrak{n})$, and prove existence results for the case that one of the Lie algebras is semisimple. For semisimple \mathfrak{g} and solvable \mathfrak{n} we show that there exist no post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$. For semisimple \mathfrak{n} and certain solvable \mathfrak{g} we construct canonical post-Lie algebra structures. On the other hand we prove that there are no post-Lie algebra structures for semisimple \mathfrak{n} and solvable, unimodular \mathfrak{g} . We also determine the generalized (α, β, γ) -derivations of \mathfrak{n} in the semisimple case. As an application we classify post-Lie algebra structures induced by generalized derivations.

1. INTRODUCTION

Post-Lie algebras and post-Lie algebra structures recently have been introduced in connection with homology of partition posets and the study of Koszul operads [19], [16]. Surprisingly they also appeared in a quite different context as well, namely in connection with nil-affine actions of Lie groups [7]. Post-Lie algebra structures generalize both LR-structures and pre-Lie algebra structures, which play a role in many areas, see [1],[3],[5],[6]. In fact, a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ with abelian Lie algebra \mathfrak{n} is just a pre-Lie algebra structure on \mathfrak{g} , and a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ with abelian Lie algebra \mathfrak{g} is an LR-structure on \mathfrak{n} . In these extreme cases it is well known that there are no such pre-Lie or LR-structures for semisimple Lie algebras. Hence a natural question is whether there exist post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$ if one of the two Lie algebras is semisimple. Certainly there are such structures if both \mathfrak{g} and \mathfrak{n} are semisimple, e.g., if they are isomorphic. Therefore one might ask, whether it is possible that a pair $(\mathfrak{g}, \mathfrak{n})$ admits a post-Lie structure, if one of the Lie algebras is semisimple, and the other is solvable. If \mathfrak{g} is semisimple, and \mathfrak{n} is solvable, it turns out that it is not possible. A proof is given in section 4. It uses the fact that a complex semisimple Lie algebra \mathfrak{g} does not admit a pre-Lie algebra structure. Conversely, if \mathfrak{n} is semisimple we construct canonical post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$ for some solvable, non-nilpotent Lie algebra \mathfrak{g} . We consider the case where \mathfrak{n} is semisimple and \mathfrak{g} is solvable and unimodular, i.e., all adjoint operators have trace zero. We show that in this case there exist no post-Lie algebra structures. In section 5 we study generalized Lie algebra derivations. A particular type, (α, β, γ) -derivations, has been studied in connection with degeneration theory of algebras, see [2],[12]. We determine the spaces of (α, β, γ) -derivations for simple Lie algebras and study post-Lie algebra structures with semisimple \mathfrak{n} , induced by such generalized derivations.

Date: August 31, 2011.

2000 *Mathematics Subject Classification.* Primary 17B30, 17D25.

The first author was supported by the FWF, Projekt P21683.

The second author expresses his gratitude towards the Erwin Schrödinger International Institute for Mathematical Physics.

2. POST-LIE ALGEBRA STRUCTURES

Let $(\mathfrak{g}, [\cdot, \cdot])$ and $(\mathfrak{n}, \{\cdot, \cdot\})$ be two Lie brackets on the same vector space V over a field k . We call $(\mathfrak{g}, \mathfrak{n})$ a *pair* of Lie algebras over k . In particular we have $\dim(\mathfrak{g}) = \dim(\mathfrak{n})$. We define a post-Lie algebra structure as follows, see [7]:

Definition 2.1. Let $(\mathfrak{g}, [\cdot, \cdot])$ and $(\mathfrak{n}, \{\cdot, \cdot\})$ be two Lie brackets on a vector space V . A *post-Lie algebra structure* on the pair $(\mathfrak{g}, \mathfrak{n})$ is a k -bilinear product $x \cdot y$ satisfying the identities:

- (1) $x \cdot y - y \cdot x = [x, y] - \{x, y\}$
- (2) $[x, y] \cdot z = x \cdot (y \cdot z) - y \cdot (x \cdot z)$
- (3) $x \cdot \{y, z\} = \{x \cdot y, z\} + \{y, x \cdot z\}$

for all $x, y, z \in V$.

The left multiplications of the algebra $A = (V, \cdot)$ are denoted by $L(x)$, i.e., we have $L(x)(y) = x \cdot y$ for all $x, y \in V$. Similarly the right multiplications $R(x)$ are given by $R(x)(y) = y \cdot x$ for all $x, y \in V$. We note two immediate consequences of the axioms [7]:

Lemma 2.2. *The map $L: \mathfrak{g} \rightarrow \text{End}(V)$ given by $x \mapsto L(x)$ is a linear representation of the Lie algebra \mathfrak{g} . Furthermore all operators $L(x)$ are Lie algebra derivations of \mathfrak{n} .*

Lemma 2.3. *Let $x \cdot y$ be a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$. Then we have the following identities:*

- (4) $x \cdot \{y, z\} + y \cdot \{z, x\} + z \cdot \{x, y\} = \{[x, y], z\} + \{[y, z], x\} + \{[z, x], y\}$
- (5) $\{x, y\} \cdot z + \{y, z\} \cdot x + \{z, x\} \cdot y = \{[x, y], z\} + \{[y, z], x\} + \{[z, x], y\} \\ + [\{x, y\}, z] + [\{y, z\}, x] + [\{z, x\}, y]$

for all $x, y, z \in V$.

Example 2.4. *If \mathfrak{n} is abelian, i.e., if $\{x, y\} = 0$ for all $x, y \in V$, then the conditions reduce to*

$$\begin{aligned} x \cdot y - y \cdot x &= [x, y], \\ [x, y] \cdot z &= x \cdot (y \cdot z) - y \cdot (x \cdot z), \end{aligned}$$

i.e., $x \cdot y$ is a pre-Lie algebra structure on the Lie algebra \mathfrak{g} .

There is a large literature on pre-Lie algebras, also called left-symmetric algebras, see [1] for a survey. It is well known that a semisimple Lie algebra does not admit a pre-Lie algebra structure. It will be useful to repeat the proof.

Proposition 2.5. *Let \mathfrak{g} be a semisimple Lie algebra over a field k of characteristic zero. Then \mathfrak{g} does not admit a pre-Lie algebra structure.*

Proof. We assume that \mathfrak{g} admits a pre-Lie algebra structure $x \cdot y$. By lemma 2.2 we have $[L(x), L(y)] = L([x, y])$ for all $x, y \in \mathfrak{g}$. Since \mathfrak{g} is semisimple we also have $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. These two conditions imply that $\text{tr}(L(x)) = 0$ for all $x \in \mathfrak{g}$. In the same way we obtain $\text{tr}(\text{ad}(x)) = 0$ for the adjoint operators. The condition $x \cdot y - y \cdot x = [x, y]$ says that $\text{ad}(x) = L(x) - R(x)$ so that $\text{tr}(R(x)) = 0$ for all $x \in \mathfrak{g}$. It also says that $\text{id} \in Z^1(\mathfrak{g}, \mathfrak{g}_L)$, where \mathfrak{g}_L denotes the \mathfrak{g} -module given by L . By Whitehead's lemma, id is a 1-coboundary. Hence there exists an element $e \in \mathfrak{g}$ with $R(e) = \text{id}$. Taking traces we obtain that the identity map has trace zero. Since the field k has characteristic zero, and $\mathfrak{g} \neq 0$, this is a contradiction. \square

Example 2.6. *If \mathfrak{g} is abelian, then the conditions reduce to*

$$\begin{aligned} x \cdot y - y \cdot x &= -\{x, y\} \\ x \cdot (y \cdot z) &= y \cdot (x \cdot z), \\ (x \cdot y) \cdot z &= (x \cdot z) \cdot y, \end{aligned}$$

i.e., $-x \cdot y$ is an LR-structure on the Lie algebra \mathfrak{n} .

For results on LR-structures on Lie algebras see [4], [5], [6]. In particular we know that a semisimple Lie algebra does not admit an LR-structure, see [5].

Proposition 2.7. *Let \mathfrak{n} be a Lie algebra over a field of characteristic zero admitting an LR-structure. Then \mathfrak{n} is solvable of class at most 2.*

Recall that we have a correspondence between post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$ and embeddings $\mathfrak{g} \hookrightarrow \mathfrak{n} \rtimes \text{Der}(\mathfrak{n})$, see [7].

Proposition 2.8. *Let $x \cdot y$ be a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$. Then the map*

$$\varphi: \mathfrak{g} \rightarrow \mathfrak{n} \rtimes \text{Der}(\mathfrak{n}), \quad x \mapsto (x, L(x))$$

is an injective homomorphism of Lie algebras. Conversely any such embedding, with the identity map on the first factor yields a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$.

Here the bracket of $\mathfrak{n} \rtimes \text{Der}(\mathfrak{n})$ is given by

$$[(x, D), (x', D')] = (\{x, x'\} + D(x') - D'(x), [D, D']).$$

Suppose that $x \cdot y$ is a post-Lie algebra structure on the pair of Lie algebras $(\mathfrak{g}, \mathfrak{n})$ such that \mathfrak{n} is centerless and satisfies $\text{Der}(\mathfrak{n}) = \text{ad}(\mathfrak{n})$. Then there is a $\varphi \in \text{End}(V)$ such that $x \cdot y = \{\varphi(x), y\}$ for all $x, y \in V$, see [7]. This means $L(x) = \text{ad}(\varphi(x))$ for the linear operators $L(x)$. In this case the axioms of a post-Lie algebra structure can be formulated as follows, see [7]:

Proposition 2.9. *Let \mathfrak{n} be a semisimple Lie algebra and $\varphi \in \text{End}(V)$. Then the product $x \cdot y = \{\varphi(x), y\}$ is a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ if and only if*

$$(6) \quad \{\varphi(x), y\} + \{x, \varphi(y)\} = [x, y] - \{x, y\},$$

$$(7) \quad \varphi([x, y]) = \{\varphi(x), \varphi(y)\}$$

for all $x, y \in V$.

The second condition says that $\varphi: \mathfrak{g} \rightarrow \mathfrak{n}$ is a Lie algebra homomorphism.

Example 2.10. *Let \mathfrak{n} be a semisimple Lie algebra. Then there are two obvious post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$ given by $\varphi = 0$ or $\varphi = -\text{id}$. In both cases \mathfrak{g} is isomorphic to \mathfrak{n} .*

Indeed, if $\varphi = 0$, the product $x \cdot y$ is zero, and $[x, y] = \{x, y\}$. In the second case, where $\varphi = -\text{id}$, we have $x \cdot y = [x, y] = -\{x, y\}$.

If \mathfrak{g} and \mathfrak{n} are both simple, these are the only possibilities. We recall the following result [7]:

Proposition 2.11. *Let $x \cdot y = \{\varphi(x), y\}$ be a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$, where \mathfrak{n} and \mathfrak{g} are simple. Then either $\varphi = 0$ or $\varphi = -\text{id}$. In both cases, \mathfrak{g} and \mathfrak{n} are isomorphic.*

The case where \mathfrak{g} and \mathfrak{n} are both semisimple is much more interesting. Suppose that \mathfrak{g} and \mathfrak{n} are isomorphic. Then there exist the obvious post-Lie structures arising from the simple

factors. On the other hand, one can find much more post-Lie algebra structures. We consider the following example. Let $\mathfrak{n} = \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$, with basis $e_1, f_1, h_1, e_2, f_2, h_2$ and Lie brackets

$$\begin{aligned} \{e_1, f_1\} &= h_1, & \{e_2, f_2\} &= h_2, \\ \{e_1, h_1\} &= -2e_1, & \{e_2, h_2\} &= -2e_2, \\ \{f_1, h_1\} &= 2f_1, & \{f_2, h_2\} &= 2f_2. \end{aligned}$$

Now define $\varphi = \left(\begin{array}{c|c} 0 & 0 \\ \hline A & 0 \end{array} \right)$ by

$$A = \begin{pmatrix} 4 & -1 & -4 \\ -1 & 1 & 2 \\ -2 & 1 & 3 \end{pmatrix}.$$

Example 2.12. *The product $x \cdot y = \{\varphi(x), y\}$ with φ given as above defines a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$, where \mathfrak{g} and \mathfrak{n} are isomorphic to $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$.*

The product is given by

$$\begin{aligned} e_1 \cdot e_2 &= -4e_2 + h_2, & f_1 \cdot e_2 &= 2e_2 - h_2, & h_1 \cdot e_2 &= 6e_2 - 2h_2, \\ e_1 \cdot f_2 &= 4f_2 + 4h_2, & f_1 \cdot f_2 &= -2f_2 - h_2, & h_1 \cdot f_2 &= -6f_2 - 4h_2, \\ e_1 \cdot h_2 &= -8e_2 - 2f_2, & f_1 \cdot h_2 &= 2e_2 + 2f_2, & h_1 \cdot h_2 &= 8e_2 + 4f_2. \end{aligned}$$

An easy calculation shows that this defines a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$, where \mathfrak{g} is given by

$$\begin{aligned} [e_1, f_1] &= h_1, & [f_1, h_1] &= 2f_1, & [h_1, f_2] &= -6f_2 - 4h_2, \\ [e_1, h_1] &= -2e_1, & [f_1, e_2] &= 2e_2 - h_2, & [h_1, h_2] &= 8e_2 + 4f_2, \\ [e_1, e_2] &= -4e_2 + h_2, & [f_1, f_2] &= -2f_2 - h_2, & [e_2, f_2] &= h_2, \\ [e_1, f_2] &= 4f_2 + 4h_2, & [f_1, h_2] &= 2e_2 + 2f_2, & [e_2, h_2] &= -2e_2, \\ [e_1, h_2] &= -8e_2 - 2f_2, & [h_1, e_2] &= 6e_2 - 2h_2, & [f_2, h_2] &= 2f_2. \end{aligned}$$

Furthermore, \mathfrak{g} is isomorphic to $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$.

Remark 2.13. It is possible to compute all post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$, where \mathfrak{n} is $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$ and \mathfrak{g} is unimodular. The result is a list of products depending on parameters which have to satisfy certain polynomial equations. It turns out that \mathfrak{g} is isomorphic to \mathfrak{n} in all cases. The matrix φ has one of the following forms:

$$\left(\begin{array}{c|c} 0 & 0 \\ \hline A & 0 \end{array} \right), \quad \left(\begin{array}{c|c} -\text{id} & 0 \\ \hline A & 0 \end{array} \right), \quad \left(\begin{array}{c|c} 0 & 0 \\ \hline A & -\text{id} \end{array} \right), \quad \left(\begin{array}{c|c} -\text{id} & 0 \\ \hline A & -\text{id} \end{array} \right),$$

and the transposed block types.

We give an example which generalizes 2.12 as follows. Let $\varphi = \left(\begin{array}{c|c} 0 & 0 \\ \hline A & 0 \end{array} \right)$ with

$$A = \begin{pmatrix} \alpha & -\frac{\beta^2}{4\alpha} & \beta \\ \gamma & -\frac{\delta^2}{4\gamma} & \delta \\ -\frac{\varepsilon}{2} & -\frac{\beta\delta}{2\varepsilon} & 1 - \frac{\beta\delta}{2\alpha} \end{pmatrix},$$

where the parameters $\alpha, \beta, \gamma, \delta, \varepsilon$ have to satisfy $\alpha\gamma \neq 0$, $\alpha\delta - \beta\gamma \neq 0$, and the polynomial conditions $\varepsilon = \alpha\delta - \beta\gamma$, $\varepsilon^2 + 4\alpha\gamma = 0$. For the choice $(\alpha, \beta, \gamma, \delta, \varepsilon) = (4, -4, -1, 2, 4)$ we obtain example 2.12.

Finally we give an easy way to construct post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$. We use a direct vector space decomposition of \mathfrak{n} by subalgebras \mathfrak{a} and \mathfrak{b} as follows.

Proposition 2.14. *Let $(\mathfrak{n}, \{, \})$ be a Lie algebra which is a direct vector space sum $\mathfrak{n} = \mathfrak{a} \oplus \mathfrak{b}$ of two subalgebras \mathfrak{a} and \mathfrak{b} . Let $(\mathfrak{g}, [,])$ be the Lie algebra given by $\mathfrak{a} \oplus \mathfrak{b}$ with the bracket*

$$[a + b, a' + b'] = \{a, a'\} - \{b, b'\}.$$

Then we obtain a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ by

$$(a + b) \cdot (a' + b') = -\{b, a' + b'\}.$$

Proof. If we view the vector space $\mathfrak{a} \oplus \mathfrak{b}$ as a Lie algebra given by the direct Lie algebra sum of \mathfrak{a} and \mathfrak{b} , then the map $\psi: \mathfrak{g} \rightarrow \mathfrak{a} \oplus \mathfrak{b}$, $\psi(a + b) = a - b$ is an algebra isomorphism. This shows that $(\mathfrak{g}, [,])$ is indeed a Lie algebra. It is different from \mathfrak{n} in general, because \mathfrak{n} is not necessarily a direct Lie algebra sum of \mathfrak{a} and \mathfrak{b} .

Let $x = a + b$, $y = a' + b'$ in $\mathfrak{a} \oplus \mathfrak{b}$. We have

$$\begin{aligned} x \cdot y - y \cdot x &= (a + b) \cdot (a' + b') - (a' + b') \cdot (a + b) \\ &= -\{b, a' + b'\} + \{b', a + b\} \\ &= \{a', b\} + 2\{b', b\} + \{b', a\} \\ &= \{a, a'\} - \{b, b'\} - \{a + b, a' + b'\} \\ &= [a + b, a' + b'] - \{a + b, a' + b'\} \\ &= [x, y] - \{x, y\}. \end{aligned}$$

This shows axiom (1). Furthermore we have

$$\begin{aligned} [x, y] \cdot z &= (\{a, a'\} - \{b, b'\}) \cdot z \\ &= \{\{b, b'\}, z\} \\ &= \{b, \{b', z\}\} - \{b', \{b, z\}\} \\ &= -\{b, b' \cdot z\} + \{b', b \cdot z\} \\ &= (a + b) \cdot ((a' + b') \cdot z) - (a' + b') \cdot ((a + b) \cdot z) \\ &= x \cdot (y \cdot z) - y \cdot (x \cdot z). \end{aligned}$$

This shows axiom (2). Finally we have

$$\begin{aligned} x \cdot \{y, z\} &= -\{b, \{y, z\}\} \\ &= -\{\{z, b\}, y\} - \{\{b, y\}, z\} \\ &= \{y, -\{b, z\}\} + \{-\{b, y\}, z\} \\ &= \{y, x \cdot z\} + \{x \cdot y, z\}, \end{aligned}$$

which shows (3). □

3. PAIRS $(\mathfrak{g}, \mathfrak{n})$ WITH SEMISIMPLE \mathfrak{n}

In this section we study post-Lie algebra structures on pairs $(\mathfrak{g}, \mathfrak{n})$, where \mathfrak{n} is semisimple. First we will use Proposition 2.14 to show that there exist canonical post-Lie algebra structures such pairs $(\mathfrak{g}, \mathfrak{n})$ where \mathfrak{n} is semisimple, or even simple, and \mathfrak{g} is some solvable Lie algebra. Suppose that \mathfrak{n} is a complex, semisimple Lie algebra with Cartan subalgebra \mathfrak{h} and a root system

of \mathfrak{n} with respect to \mathfrak{h} . If \mathfrak{n}^+ denotes the sum of the positive root spaces of \mathfrak{g} and \mathfrak{n}^- denotes the sum of the negative root spaces of \mathfrak{g} , we obtain the so called *triangular decomposition* of \mathfrak{n} ,

$$\mathfrak{n} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+,$$

which is a vector space direct sum of Lie subalgebras, where $\mathfrak{b}^+ = \mathfrak{h} \oplus \mathfrak{n}^+$ and $\mathfrak{b}^- = \mathfrak{h} \oplus \mathfrak{n}^-$ are Borel subalgebras of \mathfrak{n} , and \mathfrak{n}^\pm is nilpotent.

As an example, a triangular decomposition of $\mathfrak{n} = \mathfrak{sl}_n(\mathbb{C})$ is given by

$$\begin{aligned} \mathfrak{sl}_n(\mathbb{C}) &= \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+ \\ &= \bigoplus_{i>j} \mathbb{C}E_{ij} \oplus \bigoplus_{1 \leq i \leq n-1} \mathbb{C}(E_{ii} - E_{i+1,i+1}) \oplus \bigoplus_{i<j} \mathbb{C}E_{ij}, \end{aligned}$$

where E_{ij} is the $(n \times n)$ -matrix with entry 1 at position (i, j) and zero entries otherwise. Here \mathfrak{n}^- resp. \mathfrak{n}^+ consists of strictly lower-triangular (resp. upper-triangular) matrices.

Proposition 3.1. *Let \mathfrak{n} be a semisimple Lie algebra. Then there is a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ for some solvable non-nilpotent Lie algebra \mathfrak{g} .*

Proof. Consider a triangular decomposition $\mathfrak{n} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. We can write the direct vector space decomposition $\mathfrak{n} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ as $\mathfrak{n} = \mathfrak{a} \oplus \mathfrak{b}$ such that one algebra is nilpotent and the other one is solvable and non-nilpotent. We may choose $\mathfrak{a} = \mathfrak{n}^-$, $\mathfrak{b} = \mathfrak{b}^+$, or $\mathfrak{a} = \mathfrak{b}^-$, $\mathfrak{b} = \mathfrak{n}^+$, or interchange the roles of \mathfrak{a} and \mathfrak{b} . Then we obtain a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ by Proposition 2.14. Here \mathfrak{g} is isomorphic to a direct sum of a solvable and a nilpotent Lie algebra. Hence it is solvable. Since \mathfrak{b}^\pm is non-nilpotent, \mathfrak{g} is non-nilpotent. \square

We know from section 2 that there exist many post-Lie algebra structures on pairs $(\mathfrak{g}, \mathfrak{n})$, where both \mathfrak{n} and \mathfrak{g} are semisimple. It is perhaps less obvious that there exist post-Lie algebra structures on pairs $(\mathfrak{g}, \mathfrak{n})$, where \mathfrak{n} is semisimple, but \mathfrak{g} is not. The above Proposition shows that this is possible. If \mathfrak{n} is semisimple, \mathfrak{g} can be solvable. On the other hand, \mathfrak{g} cannot be nilpotent in that case because of the following result from [7].

Proposition 3.2. *Suppose that there is a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$, where \mathfrak{g} is nilpotent. Then \mathfrak{n} must be solvable.*

Note that the solvable Lie algebra \mathfrak{g} we obtained in Proposition 3.1 is certainly not nilpotent, since it is not even unimodular. Indeed, for semisimple \mathfrak{n} we can generalize Proposition 3.2 to the unimodular case.

Theorem 3.3. *Let \mathfrak{n} be a semisimple Lie algebra, and \mathfrak{g} be a solvable and unimodular Lie algebra. Then there is no post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$.*

Proof. Suppose there is a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$. By [7, Proposition 2.16], this means that we can view \mathfrak{g} as a subalgebra of $\mathfrak{n} \times \mathfrak{n}$, for which the map $(p_1 - p_2)|_{\mathfrak{g}}: \mathfrak{g} \rightarrow \mathfrak{n}$ is bijective. Here p_1 denotes the projection onto the first factor of $\mathfrak{n} \times \mathfrak{n}$, and p_2 the projection onto the second one. We remark here that in [7], we used the notation $\mathfrak{n} \oplus \mathfrak{n}$ in stead of $\mathfrak{n} \times \mathfrak{n}$. However, in the sequel of this proof, we will also be dealing with internal direct sums and to avoid confusion, we will reserve the symbol \oplus only for these internal sums. By assumption $p_1(\mathfrak{g})$ and $p_2(\mathfrak{g})$ are solvable, so that there exist Borel subalgebras \mathfrak{b}_1 and \mathfrak{b}_2 of \mathfrak{n} with $\mathfrak{b}_1 \supseteq p_1(\mathfrak{g})$ and $\mathfrak{b}_2 \supseteq p_2(\mathfrak{g})$. Now the intersection of two Borel subalgebras always contains a Cartan subalgebra \mathfrak{h} , and the dimension of all Borel subalgebras in \mathfrak{n} is equal to $(\dim(\mathfrak{n}) + \dim(\mathfrak{h}))/2$. For a

reference see Proposition 29.4.9 in [17]. Hence let $\mathfrak{h} \subseteq \mathfrak{b}_1 \cap \mathfrak{b}_2$ be a Cartan subalgebra of \mathfrak{n} of dimension ℓ . We have

$$\dim(\mathfrak{b}_1) = \dim(\mathfrak{b}_2) = \frac{1}{2}(\dim(\mathfrak{n}) + \ell).$$

However, since $\mathfrak{b}_1 + \mathfrak{b}_2 \supseteq p_1(\mathfrak{g}) + p_2(\mathfrak{g}) = \mathfrak{n}$ and $\mathfrak{b}_1, \mathfrak{b}_2 \subseteq \mathfrak{n}$ we also have

$$\dim(\mathfrak{b}_1) + \dim(\mathfrak{b}_2) - \dim(\mathfrak{b}_1 \cap \mathfrak{b}_2) = \dim(\mathfrak{n}).$$

The two conditions imply $\dim(\mathfrak{b}_1 \cap \mathfrak{b}_2) = \ell$. This means that $\mathfrak{b}_1 \cap \mathfrak{b}_2$ itself is a Cartan subalgebra of \mathfrak{n} . Using the root space decomposition we obtain that $\mathfrak{b}_1 = \mathfrak{a}_1 \oplus \mathfrak{h}$ and $\mathfrak{b}_2 = \mathfrak{a}_2 \oplus \mathfrak{h}$ with nilpotent ideals \mathfrak{a}_1 and \mathfrak{a}_2 . It follows that $\mathfrak{n} = \mathfrak{a}_1 \oplus \mathfrak{h} \oplus \mathfrak{a}_2$ and

$$\mathfrak{g} \leq (\mathfrak{a}_1 \oplus \mathfrak{h}) \times (\mathfrak{a}_2 \oplus \mathfrak{h}) \leq \mathfrak{n} \times \mathfrak{n}.$$

We will show that $\dim(\mathfrak{g} \cap (\mathfrak{h} \times \mathfrak{h})) = \ell$. To see this, write $x = (a_1 + h_1, a_2 + h_2) \in \mathfrak{g}$ with $a_i \in \mathfrak{a}_i, h_i \in \mathfrak{h}$ for $i = 1, 2$. Then $(p_1 - p_2)(x) = a_1 - a_2 + h_1 - h_2 \in \mathfrak{h}$ if and only if $x \in \mathfrak{h} \times \mathfrak{h}$. In this case $a_1 = a_2 = 0$ and $x = (h_1, h_2) \in \mathfrak{g} \cap (\mathfrak{h} \times \mathfrak{h})$. It follows that the map

$$(p_1 - p_2)|_{\mathfrak{g} \cap (\mathfrak{h} \times \mathfrak{h})} : \mathfrak{g} \cap (\mathfrak{h} \times \mathfrak{h}) \rightarrow \mathfrak{h}$$

is bijective. Denote by $\text{ad}_{\mathfrak{n}}(x)$ the adjoint operators of \mathfrak{n} , and by $\text{ad}_{\mathfrak{g}}(x)$ the adjoint operators of \mathfrak{g} . For any $h \in \mathfrak{h}$, we have that $\text{ad}_{\mathfrak{n}}(h)(\mathfrak{a}_i) \subseteq \mathfrak{a}_i$ ($i = 1, 2$) and $\text{ad}_{\mathfrak{n}}(h)(\mathfrak{h}) = 0$. As \mathfrak{n} is semisimple, $\text{tr ad}_{\mathfrak{n}}(h) = 0$ and hence we find that

$$(8) \quad 0 = \text{tr ad}_{\mathfrak{n}}(h) = \text{tr ad}_{\mathfrak{n}}(h)|_{\mathfrak{a}_1} + \text{tr ad}_{\mathfrak{n}}(h)|_{\mathfrak{a}_2}$$

Now, consider any $x = (h_1, h_2) \in \mathfrak{g} \cap (\mathfrak{h} \times \mathfrak{h})$ and let $y = (a_1 + \tilde{h}_1, a_2 + \tilde{h}_2)$ be any element of \mathfrak{g} . Then

$$\text{ad}_{\mathfrak{g}}(x)(y) = (\text{ad}_{\mathfrak{n}}(h_1)(a_1), \text{ad}_{\mathfrak{n}}(h_2)(a_2)).$$

From this (e.g. using explicit bases for the Lie algebras $\mathfrak{a}_1, \mathfrak{a}_2$ and \mathfrak{h}) and the assumption that \mathfrak{g} is unimodular, it follows that

$$\begin{aligned} 0 &= \text{tr ad}_{\mathfrak{g}}(x) = \text{tr ad}_{\mathfrak{n}}(h_1)|_{\mathfrak{a}_1} + \text{tr ad}_{\mathfrak{n}}(h_2)|_{\mathfrak{a}_2} \\ &= \text{tr ad}_{\mathfrak{n}}(h_1)|_{\mathfrak{a}_1} - \text{tr ad}_{\mathfrak{n}}(h_2)|_{\mathfrak{a}_1} \quad (\text{use (8)}) \\ &= \text{tr ad}_{\mathfrak{n}}(h_1 - h_2)|_{\mathfrak{a}_1}. \end{aligned}$$

As the elements $h_1 - h_2 = (p_1 - p_2)(x)$ are in 1-1 correspondence with the elements $h \in \mathfrak{h}$, it follows that $\text{tr ad}_{\mathfrak{n}}(h)|_{\mathfrak{a}_1} = 0$ for all $h \in \mathfrak{h}$. We have $\mathfrak{b}_1 = \mathfrak{h} \oplus \mathfrak{a}_1$, so that $\text{tr ad}_{\mathfrak{n}}(h)|_{\mathfrak{b}_1} = 0$ for all $h \in \mathfrak{h}$. But then the Borel subalgebra \mathfrak{b}_1 is unimodular. However, as the Borel subalgebra \mathfrak{b}_1 can always be interpreted as being of the form $\mathfrak{b}_1 = \mathfrak{h} \oplus \mathfrak{n}^+$, where \mathfrak{n}^+ is the sum of positive root spaces, there exist an $h \in \mathfrak{b}_1$ for which $\text{ad}_{\mathfrak{n}}(h)|_{\mathfrak{b}_1}$ only has non-negative (and at least one non-zero) real eigenvalues, which implies $\text{tr ad}_{\mathfrak{n}}(h)|_{\mathfrak{b}_1} > 0$, a contradiction. \square

We close this section with some lemmas on eigenspaces for semisimple derivations of \mathfrak{n} , which we will need later.

Lemma 3.4. *Let \mathfrak{n} be a semisimple Lie algebra and $z \in \mathfrak{n}$ an element such that the adjoint operator $\text{ad}(z)$ in \mathfrak{n} has three different eigenvalues $0, \alpha$ and β . Then $\alpha + \beta = 0$.*

Proof. By looking at the semisimple part we may assume that $\text{ad}(z)$ is diagonalizable. For a semisimple derivation $D \in \text{Der}(\mathfrak{n})$ we have the eigenspace decomposition $\mathfrak{n} = \bigoplus V_{\lambda_i}$ corresponding to the different eigenvalues $\lambda_1, \dots, \lambda_n$ of D . It holds $\{V_{\lambda_i}, V_{\lambda_j}\} \subseteq V_{\lambda_i + \lambda_j}$. In particular, $\{V_{\lambda_i}, V_{\lambda_j}\} = 0$ if $\lambda_i + \lambda_j$ is not an eigenvalue of D . Now choose $D = \text{ad}(z)$ and denote by $V_0, V_{\alpha}, V_{\beta}$ the eigenspaces for the derivation $\text{ad}(z)$ in \mathfrak{n} corresponding to the eigenvalues $0, \alpha$

and β . By assumption we have $\mathfrak{n} = V_0 \oplus V_\alpha \oplus V_\beta$. Let $r, s \geq 1$ denote the multiplicities of the eigenvalues α and β . We have $\text{tr}(\text{ad}(z)) = 0$ since \mathfrak{n} is semisimple. This implies $r\alpha + s\beta = 0$. Hence $\alpha + \alpha$ cannot be equal to 0 or β , because $r, s \geq 1$. Therefore $\alpha + \alpha$ is not an eigenvalue of $\text{ad}(z)$, so that $\{V_\alpha, V_\alpha\} = 0$. In the same way we obtain $\{V_\beta, V_\beta\} = 0$. Assume that $\alpha + \beta \neq 0$. Then $\{V_\alpha, V_\beta\} = 0$ and $\{V_\alpha, V_0 \oplus V_\alpha \oplus V_\beta\} \subseteq V_\alpha$. Hence V_α is an ideal of \mathfrak{n} . We have $\text{ad}(z)|_{V_\alpha} = \alpha \cdot \text{id}$. On the other hand, the trace of $\text{ad}(z)$ restricted to an ideal of \mathfrak{n} is zero. Since $\alpha \neq 0$ this is a contradiction. It follows $\alpha + \beta = 0$. \square

Denote by $\alpha(\mathfrak{n})$ the maximal dimension of an abelian subalgebra of \mathfrak{n} .

Lemma 3.5. *Let \mathfrak{n} be a semisimple Lie algebra satisfying $\alpha(\mathfrak{n}) \geq \frac{1}{3} \dim(\mathfrak{n})$. Then equality holds and \mathfrak{n} is isomorphic to $\mathfrak{sl}_2(\mathbb{C}) \oplus \cdots \oplus \mathfrak{sl}_2(\mathbb{C})$, where the number of summands is $\alpha(\mathfrak{n}) = \frac{1}{3} \dim(\mathfrak{n})$.*

Proof. Let $\mathfrak{n} = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_k$ be the decomposition of \mathfrak{n} into simple summands. Since the α -invariant is additive it follows that $\alpha(\mathfrak{n}) = \alpha(\mathfrak{s}_1) + \cdots + \alpha(\mathfrak{s}_k)$. For simple Lie algebras \mathfrak{s}_i , the α -invariant is well known, see [18]. We always have $\alpha(\mathfrak{s}_1) \leq \frac{1}{3} \dim(\mathfrak{s}_1)$, and equality holds if and only if \mathfrak{s}_i is $\mathfrak{sl}_2(\mathbb{C})$. Hence $\alpha(\mathfrak{n}) \geq \frac{1}{3} \dim(\mathfrak{n})$ is only possible if $\alpha(\mathfrak{n}) = \frac{1}{3} \dim(\mathfrak{n})$ and each summand \mathfrak{s}_i equals $\mathfrak{sl}_2(\mathbb{C})$. \square

Lemma 3.6. *Let \mathfrak{n} be a semisimple Lie algebra which is a direct vector space sum of three abelian subalgebras, i.e., $\mathfrak{n} = \mathfrak{a} \oplus \mathfrak{b} \oplus \mathfrak{c}$. Then \mathfrak{n} is isomorphic to $\mathfrak{sl}_2(\mathbb{C}) \oplus \cdots \oplus \mathfrak{sl}_2(\mathbb{C})$.*

Proof. At least one of the three abelian subalgebras of \mathfrak{n} has a dimension bigger or equal than $\frac{1}{3} \dim(\mathfrak{n})$. Therefore the α -invariant of \mathfrak{n} satisfies $\alpha(\mathfrak{n}) \geq \frac{1}{3} \dim(\mathfrak{n})$. The claim follows from Lemma 3.5. \square

4. PAIRS $(\mathfrak{g}, \mathfrak{n})$ WITH SEMISIMPLE \mathfrak{g}

We study here post-Lie algebra structures on pairs $(\mathfrak{g}, \mathfrak{n})$, where \mathfrak{g} is semisimple. It turns out that there exist no such structures with a solvable Lie algebra \mathfrak{n} . For pairs $(\mathfrak{g}, \mathfrak{n})$ with \mathfrak{g} semisimple and \mathfrak{n} abelian this is Proposition 2.5.

Let M be a \mathfrak{g} -module and $m \in M$. Denote by $\text{Ann}(m) = \{x \in \mathfrak{g} \mid x.m = 0\}$ the annihilator of m in \mathfrak{g} .

Lemma 4.1. *Let \mathfrak{g} be a semisimple Lie algebra and M be a \mathfrak{g} -module with $\dim(M) \leq \dim(\mathfrak{g})$. Then $\text{Ann}(m) \neq 0$ for all $m \in M$.*

Proof. Consider the linear map $\psi: \mathfrak{g} \rightarrow M$ given by $x \mapsto x.m$. Its kernel is $\text{Ann}(m)$. If $\dim(M) < \dim(\mathfrak{g})$ then ψ is not injective and has a non-trivial kernel. If $\dim(M) = \dim(\mathfrak{g})$, then ψ is a 1-cocycle in $Z^1(\mathfrak{g}, M)$. Assume that it is bijective. Then consider the \mathfrak{g} -module \overline{M} given by $\psi^{-1}\rho\psi$, where ρ is the representation associated to M . It is easy to see that we would obtain $\text{id} \in Z^1(\mathfrak{g}, \overline{M})$, defining a pre-Lie algebra structure on \mathfrak{g} . This contradicts Proposition 2.5. It follows that $\ker(\psi) = \text{Ann}(m)$ is non-trivial. \square

The lemma is no longer true if $\dim(M) > \dim(\mathfrak{g})$.

Theorem 4.2. *Let $(\mathfrak{g}, \mathfrak{n})$ be a pair of Lie algebras, where \mathfrak{g} is semisimple and \mathfrak{n} is solvable. Then there is no post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$.*

Proof. Assume that there is a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$. Then there exists an embedding $\mathfrak{g} \hookrightarrow \mathfrak{n} \rtimes \text{Der}(\mathfrak{n})$ for which $p_{1|_{\mathfrak{g}}}: \mathfrak{g} \rightarrow \mathfrak{n}$ is a bijection, see Proposition 2.8. Here we identify \mathfrak{g} with its image in $\mathfrak{n} \rtimes \text{Der}(\mathfrak{n})$ and denote by $p_1: \mathfrak{n} \rtimes \text{Der}(\mathfrak{n}) \rightarrow \mathfrak{n}$ the projection onto the first factor. Let $p_2: \mathfrak{n} \rtimes \text{Der}(\mathfrak{n}) \rightarrow \text{Der}(\mathfrak{n})$ be the projection onto the second factor and

$\mathfrak{s} = p_2(\mathfrak{g})$. We have $\mathfrak{g} \simeq \mathfrak{s}$ and $\mathfrak{g} \leq \mathfrak{n} \rtimes \mathfrak{s}$. Both \mathfrak{g} and \mathfrak{s} are Levi subalgebras of $\mathfrak{n} \rtimes \mathfrak{s}$. By the Levi-Malcev Theorem these are conjugated by an inner automorphism of the form $\exp(\text{ad}(z))$ for some $z \in \text{nil}(\mathfrak{n})$ in the nilradical of \mathfrak{n} . Fix an element $w \in \mathfrak{n}$ such that $\mathfrak{g} = \exp(\text{ad}(w))(\mathfrak{s})$. For $x \in \mathfrak{s}$ we have

$$\text{ad}(w)(x) = \{w, x\} = -x.w,$$

where the dot denotes the action of \mathfrak{s} on \mathfrak{n} in the semidirect product $\mathfrak{n} \rtimes \mathfrak{s}$. As a vector space we may view \mathfrak{n} as an \mathfrak{s} -module. By Lemma 4.1 there exists a nonzero element $x \in \mathfrak{s}$ such that $0 = x.w$. It follows that $\text{ad}(w)(x) = 0$ and hence $\text{ad}(w)^n(x) = 0$ for all $n \geq 1$. This implies that

$$\exp(\text{ad}(w))(x) = x$$

for this nonzero $x \in \mathfrak{s}$. It follows $x \in \mathfrak{g}$ and hence $p_1(x) = 0$. This contradicts the fact that $p_1|_{\mathfrak{g}}$ is bijective. \square

5. GENERALIZED DERIVATIONS OF LIE ALGEBRAS

In the next section we will construct a special kind of post-Lie algebra structures using certain generalized Lie algebra derivations. Indeed, there are various generalizations of them in the literature. We will be concerned here with δ -derivations of Lie algebras [10], [11], [20], with generalized derivations in the sense of Leger and Luks [15], and with (α, β, γ) -derivations of Lie algebras [12], [2]. The classes of such derivations are partly overlapping, see the definitions below. We will present some results for generalized derivations of simple and semisimple Lie algebras. In particular, we will determine all spaces of (α, β, γ) -derivations for complex simple Lie algebras.

Let \mathfrak{g} be a finite-dimensional Lie algebra over a field k . Denote by $\text{End}(\mathfrak{g})$ the space of linear maps $\varphi: \mathfrak{g} \rightarrow \mathfrak{g}$. They are not necessarily Lie algebra homomorphisms.

Definition 5.1. [12] For arbitrary scalars $\alpha, \beta, \gamma \in k$ we define the vector space $\mathcal{D}(\alpha, \beta, \gamma)$ by

$$\mathcal{D}(\alpha, \beta, \gamma) = \{\varphi \in \text{End}(\mathfrak{g}) \mid \alpha\varphi([x, y]) = \beta[\varphi(x), y] + \gamma[x, \varphi(y)] \ \forall x, y \in \mathfrak{g}\}.$$

These linear maps are called (α, β, γ) -derivations of \mathfrak{g} .

Note that $\mathcal{D}(1, 1, 1) = \text{Der}(\mathfrak{g})$ is just the space of derivations of \mathfrak{g} in the ordinary sense. Furthermore, many special cases have been considered before. We list the definitions for the special cases $(\alpha, \beta, \gamma) = (1, \delta, \delta), (1, 1, 0), (0, 1, -1)$.

Definition 5.2 (Filippov). For any $\delta \in k$ define the vector space of δ -derivations by

$$\mathcal{D}(1, \delta, \delta) = \{\varphi \in \text{End}(\mathfrak{g}) \mid \varphi([x, y]) = \delta[\varphi(x), y] + \delta[x, \varphi(y)] \ \forall x, y \in \mathfrak{g}\}.$$

Definition 5.3. The space of $(1, 1, 0)$ -derivations is given by

$$\mathcal{D}(1, 1, 0) = \{\varphi \in \text{End}(\mathfrak{g}) \mid \varphi([x, y]) = [\varphi(x), y] \ \forall x, y \in \mathfrak{g}\}.$$

Note that $\varphi \in \mathcal{D}(1, 1, 0)$ also satisfies $\varphi([x, y]) = -\varphi([y, x]) = -[\varphi(y), x] = [x, \varphi(y)]$. Hence

$$\begin{aligned} \mathcal{D}(1, 1, 0) &= \{\varphi \in \text{End}(\mathfrak{g}) \mid \varphi \circ \text{ad}(x) = \text{ad}(x) \circ \varphi \ \forall x \in \mathfrak{g}\} \\ &= C(\mathfrak{g}), \end{aligned}$$

which is called the *centroid* of \mathfrak{g} .

The centroid of \mathfrak{g} is an associative subalgebra of $\text{End}(\mathfrak{g})$, such that

$$[C(\mathfrak{g}), C(\mathfrak{g})] \subseteq \text{Hom}(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}], Z(\mathfrak{g})).$$

In particular, if \mathfrak{g} is perfect or centerless, $C(\mathfrak{g})$ is commutative.

Definition 5.4. The space of $(0, 1, -1)$ -derivations is given by

$$\begin{aligned}\mathcal{D}(0, 1, -1) &= \{\varphi \in \text{End}(\mathfrak{g}) \mid [\varphi(x), y] = [x, \varphi(y)] \ \forall x, y \in \mathfrak{g}\} \\ &= QC(\mathfrak{g}),\end{aligned}$$

which is called the *quasicentroid* of \mathfrak{g} .

The quasicentroid is also a special case of the generalized Lie algebra derivations defined by Leger and Luks. The definition is as follows.

Definition 5.5 (Leger, Luks). The space of *generalized derivations* of \mathfrak{g} is given by

$$\text{GDer}(\mathfrak{g}) = \{\varphi \in \text{End}(\mathfrak{g}) \mid \exists \sigma, \tau \in \text{End}(\mathfrak{g}) \text{ with } \tau([x, y]) = [\varphi(x), y] + [x, \sigma(y)] \ \forall x, y \in \mathfrak{g}\}.$$

An important special case is the space of quasiderivations.

Definition 5.6. The space of *quasiderivations* of \mathfrak{g} is given by

$$\text{QDer}(\mathfrak{g}) = \{\varphi \in \text{End}(\mathfrak{g}) \mid \exists \tau \in \text{End}(\mathfrak{g}) \text{ with } \tau([x, y]) = [\varphi(x), y] + [x, \varphi(y)] \ \forall x, y \in \mathfrak{g}\}.$$

We have the inclusions

$$\text{ad}(\mathfrak{g}) \subseteq \text{Der}(\mathfrak{g}) \subseteq \text{QDer}(\mathfrak{g}) \subseteq \text{GDer}(\mathfrak{g}) \subseteq \text{End}(\mathfrak{g}),$$

where $\text{ad}(\mathfrak{g})$ denotes the space of inner derivations of \mathfrak{g} . Furthermore we have

$$\begin{aligned}\text{QDer}(\mathfrak{g}) + QC(\mathfrak{g}) &= \text{GDer}(\mathfrak{g}), \\ \text{Der}(\mathfrak{g}) + C(\mathfrak{g}) &\subseteq \text{QDer}(\mathfrak{g}),\end{aligned}$$

where the first equality requires k to be a field of characteristic different from 2. Leger and Luks described conditions on \mathfrak{g} which force $QC(\mathfrak{g}) = C(\mathfrak{g})$, or equivalently $\text{GDer}(\mathfrak{g}) = \text{QDer}(\mathfrak{g})$. Theorem 5.28 of [15] proves the following result. We assume that k is the field of complex numbers.

Theorem 5.7 (Leger, Luks). *Let \mathfrak{g} be a complex centerless and perfect Lie algebra. Then $QC(\mathfrak{g}) = C(\mathfrak{g})$ and $\text{GDer}(\mathfrak{g}) = \text{QDer}(\mathfrak{g})$.*

Recall that the assumptions mean $Z(\mathfrak{g}) = 0$ and $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$.

Corollary 5.8. *Let \mathfrak{g} be a complex simple Lie algebra. Then*

$$\mathcal{D}(0, 1, -1) = QC(\mathfrak{g}) = C(\mathfrak{g}) = \mathbb{C} \cdot \text{id}.$$

Proof. Indeed, by Schur's lemma we have $C(\mathfrak{g}) = \mathbb{C} \cdot \text{id}$. The claim follows immediately from Theorem 5.7. \square

If \mathfrak{g} is simple we have $\text{Der}(\mathfrak{g}) \oplus C(\mathfrak{g}) = \text{ad}(\mathfrak{g}) \oplus \mathbb{C} \cdot \text{id} \subseteq \text{QDer}(\mathfrak{g})$. The next result shows that we have equality for simple Lie algebras different from $\mathfrak{sl}_2(\mathbb{C})$, see Corollary 4.16 of [15].

Theorem 5.9 (Leger, Luks). *Suppose that \mathfrak{g} is a complex simple Lie algebra of rank at least two. Then $\text{QDer}(\mathfrak{g}) = \text{ad}(\mathfrak{g}) \oplus \mathbb{C} \cdot \text{id}$.*

In fact, for $\mathfrak{sl}_2(\mathbb{C})$ the result is different.

Proposition 5.10. *For $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ we have $\text{QDer}(\mathfrak{g}) = \text{End}(\mathfrak{g})$.*

Proof. Let (e, f, h) be the standard basis of $\mathfrak{sl}_2(\mathbb{C})$ with $[e, f] = h$, $[h, e] = 2e$ and $[h, f] = -2f$. For a given $\varphi \in \text{End}(\mathfrak{g})$ with matrix $\varphi = (x_{ij})$ define $\tau \in \text{End}(\mathfrak{g})$ by

$$\tau = \begin{pmatrix} x_{11} + x_{33} & -x_{12} & -2x_{32} \\ -x_{21} & x_{22} + x_{33} & -2x_{31} \\ -\frac{x_{23}}{2} & -\frac{x_{13}}{2} & x_{11} + x_{22} \end{pmatrix}.$$

Then it is easy to see that τ satisfies

$$\tau([a, b]) = [\varphi(a), b] + [a, \varphi(b)]$$

for all $a, b \in \mathfrak{g}$. Hence every $\varphi \in \text{End}(\mathfrak{g})$ is a quasiderivation of \mathfrak{g} . \square

Note that the space $\text{ad}(\mathfrak{g}) \oplus \mathbb{C} \cdot \text{id}$ is 4-dimensional in this case, whereas $\text{QDer}(\mathfrak{g})$ is 9-dimensional.

We want to determine now the spaces $\mathcal{D}(\alpha, \beta, \gamma)$ for complex Lie algebras. The following elementary result was shown in [12].

Proposition 5.11. *Let \mathfrak{g} be a complex Lie algebra and $\alpha, \beta, \gamma \in \mathbb{C}$. Then $\mathcal{D}(\alpha, \beta, \gamma)$ equals one of the following subspaces of $\text{End}(\mathfrak{g})$.*

- (a) $\mathcal{D}(0, 0, 0) = \text{End}(\mathfrak{g})$.
- (b) $\mathcal{D}(1, 0, 0) = \{\varphi \in \text{End}(\mathfrak{g}) \mid \varphi([\mathfrak{g}, \mathfrak{g}]) = 0\}$.
- (c) $\mathcal{D}(0, 1, -1) = \text{QC}(\mathfrak{g})$.
- (d) $\mathcal{D}(1, 1, -1) = \mathcal{D}(0, 1, -1) \cap \mathcal{D}(1, 0, 0)$.
- (e) $\mathcal{D}(\delta, 1, 1)$, $\delta \in \mathbb{C}$.
- (f) $\mathcal{D}(\delta, 1, 0) = \mathcal{D}(0, 1, -1) \cap \mathcal{D}(2\delta, 1, 1)$, $\delta \in \mathbb{C}$.

Note that $\mathcal{D}(1, 0, 0)$ is a vector space of dimension $\dim(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]) \cdot \dim(\mathfrak{g})$, and

$$\mathcal{D}(0, 1, 0) = \{\varphi \in \text{End}(\mathfrak{g}) \mid \varphi(\mathfrak{g}) \subseteq Z(\mathfrak{g})\}$$

is a vector space of dimension $\dim(Z(\mathfrak{g})) \cdot \dim(\mathfrak{g})$. The list shows that it is enough to determine the cases (c) and (e), because then all other cases are determined as well.

Corollary 5.12. *Let \mathfrak{g} be a complex, centerless and perfect Lie algebra. Then $\mathcal{D}(0, 1, -1) = \text{QC}(\mathfrak{g}) = C(\mathfrak{g})$ is a commutative, associative subalgebra of $\text{End}(\mathfrak{g})$, and*

$$\mathcal{D}(1, 0, 0) = \mathcal{D}(1, 1, -1) = \mathcal{D}(0, 1, 0) = 0.$$

Proof. By Theorem 5.7 we have $\text{QC}(\mathfrak{g}) = C(\mathfrak{g})$. This is an associative subalgebra of $\text{End}(\mathfrak{g})$ with $[C(\mathfrak{g}), C(\mathfrak{g})] \subseteq \text{Hom}(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}], Z(\mathfrak{g}))$. If \mathfrak{g} is centerless or perfect, then this space is zero. \square

It remains to consider the spaces $\mathcal{D}(\delta, 1, 1)$ for $\delta \in \mathbb{C}$. For $\delta = 0$ we obtain the following result from [15], Lemma 6.1.

Proposition 5.13. *Let \mathfrak{g} be a complex semisimple Lie algebra. Then $\mathcal{D}(0, 1, 1) = 0$.*

If $\delta \neq 0$, then $\mathcal{D}(\delta, 1, 1) = \mathcal{D}(1, \frac{1}{\delta}, \frac{1}{\delta})$, and we deal with $\frac{1}{\delta}$ -derivations in the sense of Filippov. We obtain the following result from [11], Theorem 2, and [10], Theorem 6.

Theorem 5.14. *Let \mathfrak{g} be a complex simple Lie algebra and $\delta \neq 0, 1, -1, 2$. Then $\mathcal{D}(\delta, 1, 1) = 0$. Furthermore we have $\mathcal{D}(2, 1, 1) = \mathbb{C} \cdot \text{id}$ and $\mathcal{D}(1, 1, 1) = \text{ad}(\mathfrak{g})$.*

For the case $\delta = -1$ we obtain the following result from [11], Theorem 4.

Theorem 5.15. *Let \mathfrak{g} be a complex simple Lie algebra of rank at least two. Then*

$$\mathcal{D}(-1, 1, 1) = 0.$$

As in the case of Theorem 5.9, also here the result does not hold for $\mathfrak{sl}_2(\mathbb{C})$.

Example 5.16. *For $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ with respect to the standard basis, the space $\mathcal{D}(-1, 1, 1)$ is the linear span of*

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

This is a complement of the vector space $\text{ad}(\mathfrak{g}) \oplus \mathbb{C} \cdot \text{id}$ in $\text{End}(\mathfrak{g})$, see also Proposition 5.10.

Remark 5.17. It holds $\mathcal{D}(\delta, 1, 1) \subseteq \text{QDer}(\mathfrak{g})$ for all $\delta \in \mathbb{C}$, and we can derive the previous results also from Theorem 5.9. Indeed, let $\varphi \in \mathcal{D}(\delta, 1, 1)$, and \mathfrak{g} be a simple Lie algebra of rank at least 2. Then there is a $z \in \mathfrak{g}$ and a $\lambda \in \mathbb{C}$ such that $\varphi(x) = [z, x] + \lambda x$. By assumption we have

$$\delta \cdot \varphi([x, y]) = [\varphi(x), y] + [x, \varphi(y)]$$

for all $x, y \in \mathfrak{g}$. If we substitute φ as above, this condition can be written as

$$(\delta - 1) \cdot [z, [x, y]] = (2 - \delta)\lambda \cdot [x, y]$$

for all $x, y \in \mathfrak{g}$. Now an easy case distinction shows that $\mathcal{D}(\delta, 1, 1) = 0$ for all $\delta \neq 1, 2$, and $\mathcal{D}(2, 1, 1) = \mathbb{C} \cdot \text{id}$, $\mathcal{D}(1, 1, 1) = \text{ad}(\mathfrak{g}) \simeq \mathfrak{g}$.

We summarize the results on $\mathcal{D}(\alpha, \beta, \gamma)$ for simple Lie algebras.

Proposition 5.18. *Let \mathfrak{g} be a simple complex Lie algebra of dimension $\dim(\mathfrak{g}) \geq 4$, and $\alpha, \beta, \gamma \in \mathbb{C}$. Then $\mathcal{D}(\alpha, \beta, \gamma)$ equals one of the following subspaces of $\text{End}(\mathfrak{g})$.*

- (a) $\mathcal{D}(0, 0, 0) = \text{End}(\mathfrak{g})$.
- (b) $\mathcal{D}(1, 0, 0) = 0$.
- (c) $\mathcal{D}(0, 1, -1) = \mathbb{C} \cdot \text{id}$.
- (d) $\mathcal{D}(1, 1, -1) = 0$.
- (e) $\mathcal{D}(\delta, 1, 1) = 0$ for all $\delta \neq 1, 2$; $\mathcal{D}(1, 1, 1) = \text{ad}(\mathfrak{g})$; $\mathcal{D}(2, 1, 1) = \mathbb{C} \cdot \text{id}$.
- (f) $\mathcal{D}(\delta, 1, 0) = 0$ for all $\delta \neq 1$; $\mathcal{D}(1, 1, 0) = \mathbb{C} \cdot \text{id}$.

6. POST-LIE ALGEBRA STRUCTURES INDUCED BY GENERALIZED DERIVATIONS

Let \mathfrak{n} be semisimple. Suppose that $x \cdot y$ is a post-Lie algebra structure on the pair of Lie algebras $(\mathfrak{g}, \mathfrak{n})$. Then there is a $\varphi \in \text{End}(V)$ such that $x \cdot y = \{\varphi(x), y\}$ for all $x, y \in V$. Recall that a product $x \cdot y = \{\varphi(x), y\}$ with a linear map $\varphi: \mathfrak{g} \rightarrow \mathfrak{n}$ is a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ if and only if φ satisfies the conditions (6) and (7). Condition (7) says that φ is a Lie algebra homomorphism, whereas condition (6) looks similar to a derivation condition for φ , but involves two Lie brackets. However, if we assume that the Lie bracket of \mathfrak{g} is given as a linear function of the Lie bracket of \mathfrak{n} , then φ is indeed a quasiderivation of \mathfrak{n} .

Proposition 6.1. *Let $x \cdot y$ be a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ with \mathfrak{n} semisimple, and $x \cdot y = \{\varphi(x), y\}$ for some $\varphi \in \text{End}(V)$. Assume that $[x, y] = \tau(\{x, y\})$ for some $\tau \in \text{End}(V)$. Then $\varphi \in \text{QDer}(\mathfrak{n})$.*

Proof. Since \mathfrak{n} is semisimple, it is centerless and satisfies $\text{Der}(\mathfrak{n}) = \text{ad}(\mathfrak{n})$. Condition (6) says

$$\{\varphi(x), y\} + \{x, \varphi(y)\} = (\tau - \text{id})(\{x, y\}),$$

which means $\varphi \in \text{QDer}(\mathfrak{n})$. \square

The space $\text{QDer}(\mathfrak{n})$ has been studied. For any Lie algebra \mathfrak{n} we have

$$\text{Der}(\mathfrak{n}) + C(\mathfrak{n}) \subseteq \text{QDer}(\mathfrak{n}).$$

Suppose that \mathfrak{n} is simple of rank at least 2. Then $\text{QDer}(\mathfrak{n}) = \text{Der}(\mathfrak{n}) \oplus C(\mathfrak{n}) = \text{ad}(\mathfrak{n}) \oplus \mathbb{C} \cdot \text{id}$ by Theorem 5.9. This yields the following result.

Proposition 6.2. *Suppose that $x \cdot y$ is a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$, where \mathfrak{n} is simple of rank at least 2. Assume that $[x, y] = \tau(\{x, y\})$ for some $\tau \in \text{End}(V)$. Then*

$$(9) \quad x \cdot y = \{\{z, x\}, y\} + \lambda\{x, y\}$$

for some $z \in \mathfrak{n}$ and some $\lambda \in \mathbb{C}$.

Proof. We write $x \cdot y = \{\varphi(x), y\}$ for some $\varphi \in \text{End}(V)$. By the proposition we have $\varphi \in \text{QDer}(\mathfrak{n}) = \text{ad}(\mathfrak{n}) \oplus \mathbb{C} \cdot \text{id}$. Hence there is a $z \in \mathfrak{n}$ and a $\lambda \in \mathbb{C}$ such that $\varphi(x) = \{z, x\} + \lambda x$. This shows the claim. \square

The assumption that $[x, y] = \tau(\{x, y\})$ is of course a restriction on $(\mathfrak{g}, \mathfrak{n})$. Proposition 3.1 yields obvious examples of post-Lie structures on $(\mathfrak{g}, \mathfrak{n})$ where \mathfrak{n} is simple, and the Lie brackets of \mathfrak{g} and \mathfrak{n} do not satisfy a relation as above. Then also the conclusion of Proposition 6.2 need not hold.

We present an explicit example for $\mathfrak{n} = \mathfrak{sl}_3(\mathbb{C})$. Consider the following basis:

$$\begin{aligned} e_1 &= E_{12}, \quad e_2 = E_{13}, \quad e_3 = E_{21}, \quad e_4 = E_{23}, \quad e_5 = E_{31}, \\ e_6 &= E_{32}, \quad e_7 = E_{11} - E_{22}, \quad e_8 = E_{22} - E_{33}. \end{aligned}$$

Then the Lie brackets are defined by

$$\begin{aligned} \{e_1, e_3\} &= e_7, \quad \{e_1, e_4\} = e_2, \quad \{e_1, e_5\} = -e_6, \quad \{e_1, e_7\} = -2e_1, \quad \{e_1, e_8\} = e_1, \\ \{e_2, e_3\} &= -e_4, \quad \{e_2, e_5\} = e_7 + e_8, \quad \{e_2, e_6\} = e_1, \quad \{e_2, e_7\} = -e_2, \quad \{e_2, e_8\} = -e_2, \\ \{e_3, e_6\} &= -e_5, \quad \{e_3, e_7\} = 2e_3, \quad \{e_3, e_8\} = -e_3, \\ \{e_4, e_5\} &= e_3, \quad \{e_4, e_6\} = e_8, \quad \{e_4, e_7\} = e_4, \quad \{e_4, e_8\} = -2e_4, \\ \{e_5, e_7\} &= e_5, \quad \{e_5, e_8\} = e_5, \\ \{e_6, e_7\} &= -e_6, \quad \{e_6, e_8\} = 2e_6. \end{aligned}$$

Let $\mathfrak{sl}_3(\mathbb{C}) = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ be the triangular decomposition given by $\mathfrak{n}^- = \langle e_3, e_5, e_6 \rangle$, $\mathfrak{h} = \langle e_7, e_8 \rangle$ and $\mathfrak{n}^+ = \langle e_1, e_2, e_4 \rangle$.

Example 6.3. *The triangular decomposition $\mathfrak{sl}_3(\mathbb{C}) = \mathfrak{a} \oplus \mathfrak{b}$ with $\mathfrak{a} = \mathfrak{h} \oplus \mathfrak{n}^+$ and $\mathfrak{b} = \mathfrak{n}^-$ induces a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{sl}_3(\mathbb{C}))$ which is not of the form $x \cdot y = \{\{z, x\}, y\} + \lambda\{x, y\}$ for some $z \in \mathfrak{n}$ and some $\lambda \in \mathbb{C}$.*

By Proposition 2.14 the brackets of \mathfrak{g} are given by

$$\begin{aligned} [e_1, e_4] &= e_2, \quad [e_1, e_7] = -2e_1, \quad [e_1, e_8] = e_1, \\ [e_2, e_7] &= -e_2, \quad [e_2, e_8] = -e_2, \quad [e_3, e_6] = e_5, \\ [e_4, e_7] &= e_4, \quad [e_4, e_8] = -2e_4. \end{aligned}$$

This Lie algebra is 3-step solvable and non-nilpotent. It is of dimension 8 and has a 1-dimensional center. Assume that $[x, y] = \tau(\{x, y\})$ for some map $\tau: \mathfrak{g} \rightarrow \mathfrak{n}$. Then

$$\begin{aligned} e_1 &= [e_1, e_8] = \tau(\{e_1, e_8\}) = \tau(e_1), \\ 0 &= [e_2, e_6] = \tau(\{e_2, e_6\}) = \tau(e_1), \end{aligned}$$

which is a contradiction. The post-Lie structure is given by $x \cdot y = \{\varphi(x), y\}$ where

$$\varphi = \text{diag}(0, 0, -1, 0, -1, -1, 0, 0).$$

More precisely, the non-zero products are given by

$$\begin{aligned} e_3 \cdot e_1 &= e_7, \quad e_3 \cdot e_2 = -e_4, \quad e_3 \cdot e_6 = e_5, \quad e_3 \cdot e_7 = -2e_3, \quad e_3 \cdot e_8 = e_3, \\ e_5 \cdot e_1 &= -e_6, \quad e_5 \cdot e_2 = e_7 + e_8, \quad e_5 \cdot e_4 = e_3, \quad e_5 \cdot e_7 = -e_5, \quad e_5 \cdot e_8 = -e_5, \\ e_6 \cdot e_2 &= e_1, \quad e_6 \cdot e_3 = -e_5, \quad e_6 \cdot e_4 = e_8, \quad e_6 \cdot e_7 = e_6, \quad e_6 \cdot e_8 = -2e_6. \end{aligned}$$

It can be easily checked that φ is not of the form $\text{ad}(z) + \lambda \cdot \text{id}$.

The result of Proposition 6.2 is a motivation to study post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$ of the form (9), where \mathfrak{n} is semisimple:

Theorem 6.4. *Let \mathfrak{n} be a semisimple Lie algebra which is not a direct sum of copies of $\mathfrak{sl}_2(\mathbb{C})$. Suppose that*

$$x \cdot y = \{\{z, x\}, y\} + \lambda\{x, y\}$$

is a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ for some $z \in \mathfrak{n}$, $\lambda \in \mathbb{C}$. Then either $x \cdot y = 0$ and $[x, y] = \{x, y\}$, or $x \cdot y = -\{x, y\}$ and $[x, y] = -\{x, y\}$.

Proof. We write $x \cdot y = \{\varphi(x), y\}$ with $\varphi(x) = \{z, x\} + \lambda x$. This is a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ if and only if the identities (6) and (7) are satisfied, which means

$$(10) \quad [x, y] = \{z, \{x, y\}\} + (2\lambda + 1)\{x, y\},$$

$$(11) \quad \{\{z, x\}, \{z, y\}\} = \{z, \{z, \{x, y\}\}\} + (2\lambda + 1)\{z, \{x, y\}\} + (\lambda^2 + \lambda)\{x, y\}$$

for all $x, y \in \mathfrak{n}$. Let $\text{ad}(z)(x) = \{z, x\}$ denote the adjoint operators of \mathfrak{n} . Taking $x = z$ in (11) we obtain

$$(12) \quad \text{ad}(z)^3 + (2\lambda + 1)\text{ad}(z)^2 + (\lambda^2 + \lambda)\text{ad}(z) = 0.$$

The minimal polynomial of $\text{ad}(z)$ is a divisor of

$$t^3 + (2\lambda + 1)t^2 + (\lambda^2 + \lambda)t = t(t + \lambda + 1)(t + \lambda).$$

Hence the only possible eigenvalues of $\text{ad}(z)$ are given by 0, $-\lambda - 1$ or $-\lambda$.

Case 1: Assume that $\lambda = 0$. Since \mathfrak{n} is semisimple it follows $\text{tr}(\text{ad}(z)) = 0$. Hence all eigenvalues of $\text{ad}(z)$ are zero and we have $\text{ad}(z)^m = 0$ for some $m \geq 1$. Since (12) simplifies to $\text{ad}(z)^3 = -\text{ad}(z)^2$ we obtain $\text{ad}(z)^2 = \dots = \pm \text{ad}(z)^m = 0$. This means that z is a so called *sandwich element*. By Jacobi identity we have $\text{ad}(z)\text{ad}(x)\text{ad}(z) = 0$ for all $x \in \mathfrak{n}$. This implies $(\text{ad}(z)\text{ad}(x))^2 = 0$, so that the Killing form of \mathfrak{n} satisfies $\text{tr}(\text{ad}(z)\text{ad}(x)) = 0$ for all $x \in \mathfrak{n}$. Since it is non-degenerate it follows $z = 0$. This implies $\varphi = 0$ and $x \cdot y = 0$.

Case 2: Assume that $\lambda = -1$. Because of $\text{tr}(\text{ad}(z)) = 0$ all eigenvalues of $\text{ad}(z)$ are zero. The identity (12) reduces to $\text{ad}(z)^3 = \text{ad}(z)^2$ and we obtain $z = 0$ as before. This implies $\varphi = -\text{id}$ and $x \cdot y = -\{x, y\}$.

Case 3: Assume that $\lambda \neq 0, -1$. Now we may assume that $\text{ad}(z)$ has three different eigenvalues

$0, -\lambda, -1 - \lambda$, and its minimal polynomial is given by $t(t + \lambda + 1)(t + \lambda)$ with distinct factors. Hence $\text{ad}(z)$ is diagonalizable and we are in the situation of Lemma 3.4. Therefore we have $0 = \alpha + \beta = -\lambda - 1 - \lambda$. This means $\lambda = -\frac{1}{2}$, $\alpha = -\beta = \frac{1}{2}$ and $\mathfrak{n} = V_0 \oplus V_{\frac{1}{2}} \oplus V_{-\frac{1}{2}}$. For given $x, y \in V_0$ we have $\{z, x\} = \{z, y\} = 0$, and (11) implies $(\lambda^2 + \lambda)\{x, y\} = 0$. By assumption we can conclude that $\{x, y\} = 0$. This shows that $\{V_0, V_0\} = 0$. We also know that $\{V_{\frac{1}{2}}, V_{\frac{1}{2}}\} = \{V_{-\frac{1}{2}}, V_{-\frac{1}{2}}\} = 0$ from the proof of Lemma 3.4. Hence \mathfrak{n} is the direct vector space sum of three abelian subalgebras V_0, V_α, V_β . By Lemma 3.6 we have $\mathfrak{n} \simeq \mathfrak{sl}_2(\mathbb{C}) \oplus \cdots \oplus \mathfrak{sl}_2(\mathbb{C})$, which we have excluded. \square

One may ask what happens in the remaining case where \mathfrak{n} is a direct sum of copies of $\mathfrak{sl}_2(\mathbb{C})$. It is enough to consider the case $\mathfrak{n} = \mathfrak{sl}_2(\mathbb{C})$. Let (e, f, h) be the standard basis with $\{e, f\} = h$, $\{h, e\} = 2e$ and $\{h, f\} = -2f$. Denote by $\mathfrak{r}_{3,1}(\mathbb{C})$ the solvable non-nilpotent Lie algebra with basis e_1, e_2, e_3 and brackets $[e_1, e_2] = e_2$, $[e_1, e_3] = e_3$.

Proposition 6.5. *Let $\mathfrak{n} = \mathfrak{sl}_2(\mathbb{C})$ and suppose that $x \cdot y = \{\{z, x\}, y\} + \lambda\{x, y\}$ is a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ for some $z \in \mathfrak{n}$, $\lambda \in \mathbb{C}$. Then one of the following cases occurs.*

- (a) $z = 0$, $\lambda = 0$, $x \cdot y = 0$ and $[x, y] = \{x, y\}$.
- (b) $z = 0$, $\lambda = -1$, $x \cdot y = -\{x, y\}$ and $[x, y] = -\{x, y\}$.
- (c) $z \neq 0$, $\lambda = -\frac{1}{2}$, $x \cdot y = \{\{z, x\}, y\} - \frac{1}{2}\{x, y\}$ and $[x, y] = \{\{z, x\}, y\}$. The Lie algebra \mathfrak{g} is isomorphic to $\mathfrak{r}_{3,1}(\mathbb{C})$.

Proof. The arguments in the proof of Theorem 6.4 are also valid for $\mathfrak{n} = \mathfrak{sl}_2(\mathbb{C})$. In the third case however we can classify all post-Lie algebra structures. We first conclude $\lambda = -\frac{1}{2}$ and $x \cdot y = \{\{z, x\}, y\} - \frac{1}{2}\{x, y\}$. Then, writing $z = \alpha e + \beta f + \gamma h$, an easy calculation shows that this product is a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$ if and only if

$$\frac{\alpha\beta + \gamma^2}{16} = 1.$$

In particular we have $z \neq 0$. The Lie brackets of \mathfrak{g} are given by

$$\begin{aligned} [e, f] &= -2\alpha e + 2\beta f, \\ [e, h] &= -4\gamma e + 2\beta h, \\ [f, h] &= -4\gamma f + 2\alpha h. \end{aligned}$$

This is a solvable non-nilpotent Lie algebra which is isomorphic to $\mathfrak{r}_{3,1}(\mathbb{C})$ for all α, β, γ with $\alpha\beta + \gamma^2 = 1/16$. \square

REFERENCES

- [1] D. Burde: *Left-symmetric algebras, or pre-Lie algebras in geometry and physics*. Central European Journal of Mathematics **4** (2006), no. 3, 323–357.
- [2] D. Burde, T. Beneš: *Degenerations of pre-Lie algebras*. J. Math. Phys. **50**, 112102 (2009).
- [3] D. Burde, K. Dekimpe: *Novikov structures on solvable Lie algebras*. J. Geom. Phys. **56** (2006), no. 9, 1837–1855.
- [4] D. Burde, K. Dekimpe and S. Deschamps: *Affine actions on nilpotent Lie groups*. Forum Math. **21** (2009), no. 5, 921–934.
- [5] D. Burde, K. Dekimpe and S. Deschamps: *LR-algebras*. Contemporary Mathematics **491** (2009), 125–140.
- [6] D. Burde, K. Dekimpe, K. Verkammen: *Complete LR-structures on solvable Lie algebras*. Journal of group theory **13**, Issue 5, 703–719 (2010).
- [7] D. Burde, K. Dekimpe and K. Verkammen: *Affine actions on Lie groups and post-Lie algebra structures*. Preprint (2011).

- [8] K. Dekimpe: *Semi-simple splittings for solvable Lie groups and polynomial structures*. Forum Math. **12** (2000), 77–96.
- [9] A. Dzhumadil'daev, P. Zusmanovich: *Commutative 2-cocycles on Lie algebras*. J. Algebra **324** (2010), no. 4, 732–748.
- [10] V. T. Filippov: *On δ -derivations of Lie algebras*. Siberian Mathematical Journal, Vol. 39, No. 6 (1998), 1218–1230.
- [11] V. T. Filippov: *δ -derivations of prime Lie algebras*. Siberian Mathematical Journal, Vol. 40, No. 1 (1999), 174–184.
- [12] J. Hrivnák, P. Novotný: *On (α, β, γ) -derivations of Lie algebras and corresponding invariant functions*. J. Geom. Phys. **58** (2008), 208–217.
- [13] H. Kim: *Complete left-invariant affine structures on nilpotent Lie groups*. J. Differential Geom. **24** (1986), No. 3, 373–394.
- [14] A.W. Knap: *Lie groups, Lie algebras, and cohomology*. Mathematical Notes **34** (1988) Princeton University Press, Princeton.
- [15] G. F. Leger, E. M. Luks: *Generalized Derivations of Lie Algebras*. J. Algebra **228**, (2000), no. 1, 165–203.
- [16] J.-L. Loday: *Generalized bialgebras and triples of operads*. Astisque No. **320** (2008), 116 pp.
- [17] P. Tauvel, R. W. T. Yu: *Lie algebras and algebraic groups*. Monographs in Mathematics. Springer-Verlag, Berlin (2005), 653 pp.
- [18] R. Suter: *Abelian ideals in a Borel subalgebra of a complex simple Lie algebra*. Invent. Math. **156** (2004), 175–221.
- [19] B. Vallette: *Homology of generalized partition posets*. J. Pure and Applied Algebra **208**, No. 2 (2007), 699–725.
- [20] P. Zusmanovich: *On δ -derivations of Lie algebras and superalgebras*. J. Algebra **324** (2010), no. 12, 3470–3486.

FAKULTÄT FÜR MATHEMATIK, UNIVERSITÄT WIEN, NORDBERGSTR. 15, 1090 WIEN, AUSTRIA
E-mail address: dietrich.burde@univie.ac.at

KATHOLIEKE UNIVERSITEIT LEUVEN, CAMPUS KORTRIJK, 8500 KORTRIJK, BELGIUM
E-mail address: karel.dekimpe@kuleuven-kortrijk.be